

Skew-signings of positive weighted digraphs

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Abstract

An *arc-weighted digraph* is a pair (D, ω) where D is a digraph and ω is an *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$. Given an arc-weighted digraph (D, ω) with vertices v_1, \dots, v_n , the *weighted adjacency matrix* of (D, ω) is defined as the matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$, if $v_i v_j$ an arc of D and 0 otherwise. Let (D, ω) be a positive arc-weighted digraphs and assume that D is loopless and symmetric. A *skew-signing* of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D . In this paper, we give necessary and sufficient conditions under which the characteristic polynomial of $A(D, \omega')$ is the same for every skew-signing ω' of (D, ω) . Our Main Theorem generalizes a result of Cavers et al (2012) about skew-adjacency matrices of graphs.

Keywords: Arc-weighted digraphs; Skew-signing of a digraph; Weighted adjacency matrix.

2000 MSC: 05C22, 05C31, 05C50

1. Introduction

Let G be a simple undirected and finite graph. An *orientation* of G is an assignment of a direction to each edge of G so that we obtain a directed graph \vec{G} . Let \vec{G} be an orientation of G . With respect to a labeling v_1, \dots, v_n of the vertices of G , the *skew-adjacency matrix* of \vec{G} is the real skew-symmetric

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matrix $S(\vec{G}) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ij} = -1$ if $v_i v_j$ is an arc of \vec{G} , otherwise $s_{ij} = s_{ji} = 0$. The *skew-characteristic polynomial* of \vec{G} is defined as the characteristic polynomial of $S(\vec{G})$. This definition is correct because skew-adjacency matrices of \vec{G} with respect to different labelings are permutationally similar and so have the same characteristic polynomial. There are several recent works about skew-characteristic polynomials of oriented graphs, one can see for example [1, 4, 5, 6, 10, 12]. Given a graph G , an open problem is to find the number of possible orientations of G with distinct skew-characteristic polynomials. In particular it is of interest to know whether all orientations of G can have the same skew-characteristic polynomial. The following theorem, obtained by Cavers et al. [4] gives an answer to this question.

Theorem 1.1. *The orientations of a graph G all have the same characteristic polynomial if and only if G has no cycles of even length.*

A similar result was obtained by Liu and Zhang [7]. They proved that all orientations of a graph G have the same permanent polynomial if and only if G has no cycles of even length.

In this work, we will extend Theorem 1.1 to arc-weighted digraphs. Recall that a *directed graph* or a *digraph* D is a pair $D = (V, E)$ where V is a set of *vertices* and E is a set of ordered pairs of vertices called *arcs*. For $u, v \in V$, an arc $a = (u, v)$ of D is denoted by uv . An arc of the form uu is called a *loop* of D . A *loopless digraph* is one containing no loops. A *symmetric* digraph is a digraph such that if uv is an arc then vu is also an arc. A *directed cycle* of length $t > 0$ in a digraph D is a subdigraph of D with vertex set $\{v_1, v_2, \dots, v_t\}$ and arcs $v_1 v_2, \dots, v_{t-1} v_t, v_t v_1$. Throughout the paper, we use the term "cycle" to refer to a "directed cycle" in a digraph. A cycle of length $t = 2$ is called a *digon*. A cycle is *odd* (resp. *even*) if its length is odd (resp. even).

An *arc-weighted digraph* or more simply a *weighted digraph* is a pair (D, ω) where D is a digraph and ω is a *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$, called the *weight* of the arc uv . Let (D, ω) be a weighted digraph with vertices v_1, \dots, v_n . The *weighted adjacency matrix* of (D, ω) is defined as the $n \times n$ matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$, if $v_i v_j$ is an arc of D and 0 otherwise.

Every $n \times n$ real matrix $M = [m_{ij}]$ is the weighted adjacency matrix of a unique weighted digraph (D_M, ω_M) with vertex set $\{1, \dots, n\}$. This digraph

is called the *weighted digraph associated to M* and defined as follows: ij is an arc of D_M iff $m_{ij} \neq 0$, and the weight of an arc ij is $\omega_M(ij) = m_{ij}$.

In the remainder of this paper, we consider only positive weighted loopless and symmetric digraphs (which we abbreviate to *pwls-digraphs*). Let (D, ω) be a pwls-digraphs. A *skew-signing* of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D .

The main result of this paper is the following theorem.

Theorem 1.2. *Let (D, ω) be pwls-digraph and let ω' be a skew-signing of (D, ω) . Then, the following statements are equivalent:*

- i) *The characteristic polynomial of (D, ω') is the same for any skew-signing ω' of (D, ω) .*
- ii) *D has no even cycles of length more than 2 and $A(D, \omega) = \Delta^{-1}S\Delta$ where S is a nonnegative symmetric matrix with zero diagonal and Δ is a diagonal matrix with positive diagonal entries.*

Remark that a graph G can be identified to the pwls-digraph obtained from G by replacing each edge $\{u, v\}$ by the two arcs uv and vu , both of them have weight 1. Moreover, every orientation of G can be identified to a skew-signing of this digraph. Then, our main result is a generalization of Theorem 1.1.

2. Cycle-symmetric digraphs

We start with some formulas involving the characteristic polynomial of a matrix and its weighted associated digraph. For this, we need some notations and definitions. Let D be a digraph. A *linear subdigraph* L of D is a vertex disjoint union of some cycles in D . A linear subdigraph L of D is called *even linear* if L contains no odd cycle. Let $\vec{\mathcal{L}}_k(D)$ (resp. $\vec{\mathcal{L}}_k^e(D)$) denote the set of all linear (resp. even linear) subdigraphs of D that cover precisely k vertices of D . We usually write this as $\vec{\mathcal{L}}_k$ (resp. $\vec{\mathcal{L}}_k^e$) when no ambiguity can arise.

Let A be a real matrix and let (D, ω) the weighted digraph associated to A . We denote by $p_A(x) = \det(xI - A) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ the characteristic polynomial of A .

By using the classical definition of the determinant, we obtain the following formula:

$$a_k = \sum_{\vec{L} \in \vec{\mathcal{L}}_k} (-1)^{|\vec{L}|} \omega(\vec{L}) \quad (1)$$

where $|\vec{L}|$ denotes the number of cycles in \vec{L} and $\omega(\vec{L})$ is the product of all the weights of the arcs of \vec{L} .

In particular

$$\det(A) = (-1)^n a_n = (-1)^n \sum_{\vec{L} \in \vec{\mathcal{L}}_n} (-1)^{|\vec{L}|} \omega(\vec{L}). \quad (2)$$

If A is skew-symmetric, then

$$a_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\vec{L} \in \vec{\mathcal{L}}_k^e} (-1)^{|\vec{L}|} \omega(\vec{L}) & \text{if } k \text{ is even} \end{cases} \quad (3)$$

We introduce now a special class of weighted symmetric digraphs called cycle-symmetric digraphs. The characterization of these digraphs will be used in the proof of our main theorem.

Given a symmetric digraph $D = (V, E)$ and a subdigraph $H = (W, F)$ of D , we denote by H^* the subdigraph of D whose vertex set is W and arc set is $\{vu : uv \in F\}$. Consider now a positive arc-weight function ω of D and let q be a positive integer. We say that (D, ω) is $(\leq q)$ -cycle-symmetric if $\omega(\vec{C}) = \omega(\vec{C}^*)$ for every cycle \vec{C} of D of length at most q . If $q = n$ then (D, ω) is said to be *cycle-symmetric*.

We borrowed the terminology "cycle-symmetric" from Shih and Weng [11]. Following this paper, an $n \times n$ real matrix $[a_{ij}]$ is called *cycle-symmetric* if the two conditions hold:

- C1)** for $i \neq j \in \{1, \dots, n\}$, $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$;
- C2)** For any sequence of distinct integers i_1, \dots, i_k from the set $\{1, \dots, n\}$, we have

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} = a_{i_1 i_k} a_{i_k i_{k-1}} \cdots a_{i_3 i_2} a_{i_2 i_1}$$

Obviously, a pwls-digraph (D, ω) is always (≤ 2) -cycle-symmetric. Moreover, a pwls-digraph is cycle-symmetric if and only if its weighted adjacency matrix is cycle-symmetric.

The following theorem gives a characterization of cycle-symmetric matrices.

Theorem 2.1. *An $n \times n$ real matrix A is cycle-symmetric if and only if there exists an invertible diagonal matrix D such that $D^{-1}AD$ is symmetric.*

Different proofs of this theorem are given in [8, 9, 11]. As a consequence, we obtain the following characterization of cycle-symmetric pwls-digraphs.

Corollary 2.2. *Let (D, ω) be a pwls-digraph. Then, the following statements are equivalent:*

- i) (D, ω) is cycle-symmetric.
- ii) $A(D, \omega) = \Delta^{-1}S\Delta$ where S is a nonnegative symmetric matrix with zero diagonal and Δ is a diagonal matrix with positive diagonal entries.

3. Skew-signings of cycle-symmetric digraphs

In this section, we study cycle-symmetric pwsl-digraphs (D, ω) such that the characteristic polynomial of (D, ω') is the same for any skew-signing ω' of (D, ω) . More precisely, we will prove the following Proposition.

Proposition 3.1. *Let (D, ω) be a cycle-symmetric pwsl-digraph. Then, the following statements are equivalent:*

- i) *The characteristic polynomial of (D, ω') is the same for any skew-signing ω' of (D, ω) ;*
- ii) *D contains no even cycle of length greater than 3.*

Let (D, ω) is an arbitrary pwls-digraph and let ω' be a skew-signing ω' of (D, ω) . We consider the two arc-weight functions defined as follows: $\overline{\omega}(uv) = \sqrt{\omega(uv)\omega(vu)}$ and $\widehat{\omega}'(uv) = \frac{\omega'(uv)}{\omega(uv)} \sqrt{\omega(uv)\omega(vu)}$ for every arc uv of D . We have the following properties:

P1 The weighted adjacency matrix of $(D, \overline{\omega})$ is symmetric.

P2 $\widehat{\omega}'$ is a skew-signing of $(D, \overline{\omega})$ and $A(D, \widehat{\omega}')$ is a skew-symmetric matrix.

P3 If (D, ω) is a $(\leq q)$ -cycle-symmetric digraph, then for every dicycle \vec{C} of D with length $\leq q$ we have $\omega(\vec{C}) = \overline{\omega}(\vec{C})$ and $\omega'(\vec{C}) = \widehat{\omega}'(\vec{C})$.

We denote by $p_{(D, \omega)}(x) := x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ the characteristic polynomial of (D, ω) . The characteristic polynomials of (D, ω') and $(D, \widehat{\omega}')$ are respectively denoted by $p_{(D, \omega')}(x) := x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n$ and $p_{(D, \widehat{\omega}')} (x) := x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n$.

From Formula (1), we have $b_1 = 0$ and $b_2 = -a_2$. In particular, b_1 and b_2 are independent of ω' .

Lemma 3.2. *If (D, ω) is $(\leq q)$ -cycle-symmetric, then:*

$$b_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\vec{L} \in \vec{\mathcal{L}}_k^e} (-1)^{|\vec{L}|} \omega'(\vec{L}) & \text{if } k \text{ is even} \end{cases}$$

for $k = 1, \dots, q$.

Proof. Let $k \in \{1, \dots, q\}$. It follows from **P3** that $\omega'(\vec{L}) = \widehat{\omega}'(\vec{L})$ for every $\vec{L} \in \vec{\mathcal{L}}_k$ and hence by using Formula (1), we have, $b_k = c_k$. Moreover, $A(D, \widehat{\omega}')$ is a skew-symmetric matrix then by formula (3):

$$c_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\vec{L} \in \vec{\mathcal{L}}_k^e} (-1)^{|\vec{L}|} \widehat{\omega}'(\vec{L}) & \text{if } k \text{ is even} \end{cases}$$

Now, by applying again **P2**, we obtain

$$b_k = c_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\vec{L} \in \vec{\mathcal{L}}_k^e} (-1)^{|\vec{L}|} \omega'(\vec{L}) & \text{if } k \text{ is even} \end{cases} \quad \square$$

We denote by $\vec{\mathcal{C}}_k$ the set of cycles of length k of D . For a skew-signing ω' , this set is partitioned into two subsets: $\vec{\mathcal{C}}_{k, \omega'}^+$ and $\vec{\mathcal{C}}_{k, \omega'}^-$ where $\vec{\mathcal{C}}_{k, \omega'}^+$ (resp. $\vec{\mathcal{C}}_{k, \omega'}^-$) is the set of cycles \vec{C} such that $\omega'(\vec{C}) > 0$ (resp. $\omega'(\vec{C}) < 0$). For k even, let $\vec{\mathcal{D}}_k$ denote the set of all collections \vec{L} of vertex disjoint digons that cover precisely k vertices in D .

Corollary 3.3. *Let $q \geq 4$. Assume that (D, ω) is $(\leq q-1)$ -cycle-symmetric and contains no even cycles of length $k \in \{3, \dots, q-1\}$ then*

$$b_k = \begin{cases} \sum_{\vec{L} \in \vec{\mathcal{D}}_k} \omega(\vec{L}) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

for $k = 1, \dots, q-1$ and

$$b_q = \begin{cases} - \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^+} \omega(\vec{C}) + \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^-} \omega(\vec{C}) + \sum_{\vec{L} \in \vec{\mathcal{D}}_q} \omega(\vec{L}) & \text{if } q \text{ is even} \\ - \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^+} \omega(\vec{C}) + \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^-} \omega(\vec{C}) & \text{if } q \text{ is odd} \end{cases}$$

Proof. The first equality follows from Lemma 3.2.

From formula (1), we have

$$\begin{aligned} b_q &= \sum_{\vec{L} \in \vec{\mathcal{L}}_q} (-1)^{|\vec{L}|} \omega'(\vec{L}) \\ &= \sum_{\vec{L} \in \vec{\mathcal{L}}_q \setminus \vec{\mathcal{L}}_q^e} (-1)^{|\vec{L}|} \omega'(\vec{L}) + \sum_{\vec{L} \in \vec{\mathcal{D}}_q} (-1)^{|\vec{L}|} \omega'(\vec{L}) - \sum_{\vec{C} \in \vec{\mathcal{C}}_q} \omega'(\vec{C}) \end{aligned}$$

By definition of $\vec{\mathcal{C}}_{q, \omega'}^+$ and $\vec{\mathcal{C}}_{q, \omega'}^-$, we have $\sum_{\vec{C} \in \vec{\mathcal{C}}_q} \omega'(\vec{C}) = \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^+} \omega(\vec{C}) - \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'}^-} \omega(\vec{C})$.

Consider now $\vec{L} \in \vec{\mathcal{L}}_q \setminus \vec{\mathcal{L}}_q^e$. By definition of $\vec{\mathcal{L}}_q$ and $\vec{\mathcal{L}}_q^e$, the linear subdigraph \vec{L} contains an odd cycle \vec{C} among its components. Let \vec{L}' the linear subdigraph obtained from \vec{L} by replacing the cycle \vec{C} by \vec{C}^* . Since \vec{C} is odd and $\omega(\vec{C}) = \omega(\vec{C}^*)$, $\omega'(\vec{L}) = -\omega'(\vec{L}')$. Thus, linear subdigraphs of $\vec{\mathcal{L}}_q \setminus \vec{\mathcal{L}}_q^e$ contribute 0 to b_p . Now, according to the parity of p , we have $\sum_{\vec{L} \in \vec{\mathcal{D}}_p} (-1)^{|\vec{L}|} \omega'(\vec{L}) = \sum_{\vec{L} \in \vec{\mathcal{D}}_p} (-1)^{p/2} (-1)^{p/2} \omega(\vec{L}) = \sum_{\vec{L} \in \vec{\mathcal{D}}_p} \omega(\vec{L})$ if p is even and 0 if p is odd, which yields the second equality in the Corollary. \square

It follows that if (D, ω) is cycle-symmetric and contains no even cycles of length greater than 2 then the characteristic polynomial of (D, ω') is the

same for any skew-signing ω' of (D, ω) . This proves the implication $ii) \Rightarrow i)$ of Proposition 3.1. The proof of $i)$ implies $ii)$ is a direct consequence of the following more general result.

Lemma 3.4. *Let (D, ω) be a $(\leq l)$ -cycle-symmetric pwsd-digraph where $l \geq 3$. If the characteristic polynomial of (D, ω') is the same for any skew-signing ω' of (D, ω) , then every cycle of length at most l is an odd cycle or a digon.*

Before proving this Lemma, we introduce some notations and establish an intermediate result. Let (D, ω) be an arbitrary pwsd-digraph and consider an arbitrary cycle of D of length $q \geq 3$ whose vertices are v_1, \dots, v_q and whose arcs are $e_1 := v_1v_2, \dots, e_{q-1} := v_{q-1}v_q, e_q := v_qv_1$. Let ω' be a skew-signing of (D, ω) . For $h \in \{1, \dots, q\}$ and $r \in \{1, \dots, h\}$, we denote by $\eta_{\omega'}^+(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h})$ the sum of the weights of cycles \vec{C} of length q in D that have $\omega'(\vec{C}) > 0$ and contain arcs e_1, \dots, e_r but not arcs e_{r+1}, \dots, e_h . Define $\eta_{\omega'}^-(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h})$ analogously.

Lemma 3.5. *There exists a skew-signing ω'_0 of (D, ω) such that $\eta_{\omega'_0}^+(e_1) \neq \eta_{\omega'_0}^-(e_1)$.*

Proof. Assume the contrary. We claim that for each $t \in \{1, \dots, q\}$ and for all skew-signing ω' of (D, ω) , $\eta_{\omega'}^+(e_1, \dots, e_t) = \eta_{\omega'}^-(e_1, \dots, e_t)$. For this, we proceed by induction on t . The case $t = 1$ is assumed. Let $t \in \{1, \dots, q-1\}$ and suppose that the claim is true for t . Then

$$\begin{cases} \eta_{\omega'}^+(e_1, \dots, e_t) = \eta_{\omega'}^+(e_1, e_2, \dots, e_t, e_{t+1}) + \eta_{\omega'}^+(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \\ \eta_{\omega'}^-(e_1, \dots, e_t) = \eta_{\omega'}^-(e_1, e_2, \dots, e_t, e_{t+1}) + \eta_{\omega'}^-(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \end{cases}$$

Consider now the skew-signing ω'' that coincides with ω' outside $\{e_{t+1}, e_{t+1}^*\}$ and such that $\omega''(e) = -\omega'(e)$ for $e \in \{e_{t+1}, e_{t+1}^*\}$.

Then, we have

$$\begin{cases} \eta_{\omega''}^+(e_1, \dots, e_t) = \eta_{\omega'}^-(e_1, e_2, \dots, e_t, e_{t+1}) + \eta_{\omega'}^+(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \\ \eta_{\omega''}^-(e_1, \dots, e_t) = \eta_{\omega'}^+(e_1, e_2, \dots, e_t, e_{t+1}) + \eta_{\omega'}^-(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \end{cases}$$

But by induction hypothesis, we have $\eta_{\omega''}^+(e_1, \dots, e_t) = \eta_{\omega''}^-(e_1, \dots, e_t)$ and $\eta_{\omega'}^+(e_1, \dots, e_t) = \eta_{\omega'}^-(e_1, \dots, e_t)$.

Then

$$\begin{aligned} \eta_{\omega''}^+(e_1, \dots, e_t, \overline{e_{t+1}}) - \eta_{\omega''}^-(e_1, \dots, e_t, \overline{e_{t+1}}) &= \eta_{\omega'}^-(e_1, \dots, e_{t+1}) - \eta_{\omega'}^+(e_1, \dots, e_{t+1}) \\ &= \eta_{\omega'}^+(e_1, \dots, e_{t+1}) - \eta_{\omega'}^-(e_1, \dots, e_{t+1}) \end{aligned}$$

Thus $\eta_{\omega'}^+(e_1, \dots, e_t, e_{t+1}) = \eta_{\omega'}^-(e_1, e_2, \dots, e_t, e_{t+1})$.

This complete the induction proof. For $t = q$ we have , $\eta_{\omega'}^+(e_1, \dots, e_q) = \eta_{\omega'}^-(e_1, \dots, e_q)$.

Now, choose a skew-signing ω' of (D, ω) such that $\omega'(e_1) = \omega(e_1), \dots, \omega'(e_q) = \omega(e_q)$. Then, we have $\eta_{\omega'}^+(e_1, \dots, e_q) = \prod_{i=1}^q \omega(e_i)$ and $\eta_{\omega'}^-(e_1, \dots, e_q) = 0$, a contradiction. It follows that there exists a skew-signing ω'_0 such that $\eta_{\omega'_0}^+(e_1) \neq \eta_{\omega'_0}^-(e_1)$. \square

PROOF OF LEMMA 3.4. Assume for contradiction that D contains an even cycle of length $q \in \{4, \dots, l\}$ and choose such a cycle with q as small as possible. We will use the notations of the previous lemma. Let ω'' be the skew-signing of (D, ω) that coincides with ω'_0 outside $\{e_1, e_1^*\}$ and such that $\omega''(e) = -\omega'_0(e)$ for $e \in \{e_1, e_1^*\}$. The charateristic polynomials of (D, ω'_0) and (D, ω'') are respectively denoted by $p_{(D, \omega'_0)}(x) := x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n$ and $p_{(D, \omega'')}(x) := x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n$.

By the choice of q and from the second equality of Corollary 3.3, we have $b_q - c_q = - \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'_0}^+} \omega(\vec{C}) + \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega'_0}^-} \omega(\vec{C}) + \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega''}^+} \omega(\vec{C}) - \sum_{\vec{C} \in \vec{\mathcal{C}}_{q, \omega''}^-} \omega(\vec{C})$.

Every cycle \vec{C} of length q that contains neither e_1 nor e_1^* contributes 0 to $b_q - c_q$. It follows that:

$$\begin{aligned} b_q - c_q &= -\eta_{\omega'_0}^+(e_1) - \eta_{\omega'_0}^+(e_1^*) + \eta_{\omega'_0}^-(e_1) + \eta_{\omega'_0}^-(e_1^*) \\ &\quad + \eta_{\omega''}^+(e_1) + \eta_{\omega''}^+(e_1^*) - \eta_{\omega''}^-(e_1) - \eta_{\omega''}^-(e_1^*) \end{aligned}$$

By construction of ω'' , we have $\eta_{\omega''}^+(e_1) = \eta_{\omega'_0}^-(e_1)$, $\eta_{\omega''}^-(e_1) = \eta_{\omega'_0}^+(e_1)$, $\eta_{\omega''}^+(e_1^*) = \eta_{\omega'_0}^-(e_1^*)$, $\eta_{\omega''}^-(e_1^*) = \eta_{\omega'_0}^+(e_1^*)$.

Then $b_q - c_q = -2(\eta_{\omega'_0}^+(e_1) + \eta_{\omega'_0}^+(e_1^*)) + 2(\eta_{\omega'_0}^-(e_1) + \eta_{\omega'_0}^-(e_1^*))$

As (D, ω) is $(\leq l)$ -cycle-symmetric, we have $\eta_{\omega'_0}^+(e_1) = \eta_{\omega'_0}^+(e_1^*)$, $\eta_{\omega'_0}^-(e_1) = \eta_{\omega'_0}^-(e_1^*)$ and then $b_q - c_q = -4(\eta_{\omega'_0}^+(e_1) - \eta_{\omega'_0}^-(e_1)) \neq 0$, a contradiction.

4. Proof of the main theorem

The implication $ii) \implies i)$ follows easily from Corollary 2.2 and Proposition 3.1. To prove $i) \implies ii)$ it suffices to use Proposition 3.1 and the next Lemma.

Lemma 4.1. *Let (D, ω) be a pwls-digraph. If the characteristic polynomial of (D, ω') is the same for any skew-signing ω' of (D, ω) , then (D, ω) is cycle-symmetric.*

Proof. Assume for contradiction that (D, ω) is not cycle-symmetric and let \vec{C}_0 be a shortest cycle of D such that $\omega(\vec{C}_0) \neq \omega(\vec{C}_0^*)$. We denote by v_1, \dots, v_q the vertices of \vec{C}_0 and $e_1 := v_1v_2, \dots, e_{q-1} := v_{q-1}v_q, e_q := v_qv_1$ its arcs. Let ω' be a skew-signing of (D, ω) .

For $h \in \{1, \dots, q\}$ and $r \in \{1, \dots, h\}$, we set

$$\begin{aligned} N_{\omega'}^+(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h}) &= \eta_{\omega'}^+(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h}) \\ &\quad + \eta_{\omega'}^+(e_1^*, \dots, e_r^*, \overline{e_{r+1}^*}, \dots, \overline{e_h^*}) \\ N_{\omega'}^-(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h}) &= \eta_{\omega'}^-(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h}) \\ &\quad + \eta_{\omega'}^-(e_1^*, \dots, e_r^*, \overline{e_{r+1}^*}, \dots, \overline{e_h^*}) \end{aligned}$$

Step 1 There exists a skew-signing ω'_0 of (D, ω) such that $N_{\omega'_0}^+(e_1) \neq N_{\omega'_0}^-(e_1)$.

Assume by contradiction that $N_{\omega'}^+(e_1) = N_{\omega'}^-(e_1)$ for every skew-signing ω' of (D, ω) . By using an induction process, we can deduce, as in the proof of Lemma 3.5, that $N_{\omega'}^+(e_1, \dots, e_q) = N_{\omega'}^-(e_1, \dots, e_q)$. However,

$$N_{\omega'}^+(e_1, \dots, e_q) = \begin{cases} \omega(\vec{C}_0) + \omega(\vec{C}_0^*) & \text{if } q \text{ is even and } \omega'(\vec{C}_0) > 0 \\ 0 & \text{if } q \text{ is even and } \omega'(\vec{C}_0) < 0 \\ \omega(\vec{C}_0) & \text{if } q \text{ is odd and } \omega'(\vec{C}_0) > 0 \\ \omega(\vec{C}_0^*) & \text{if } q \text{ is odd and } \omega'(\vec{C}_0) < 0 \end{cases}$$

and

$$N_{\omega'}^-(e_1, \dots, e_q) = \begin{cases} 0 & \text{if } q \text{ is even and } \omega'(\vec{C}_0) > 0 \\ \omega(\vec{C}_0) + \omega(\vec{C}_0^*) & \text{if } q \text{ is even and } \omega'(\vec{C}_0) < 0 \\ \omega(\vec{C}_0^*) & \text{if } q \text{ is odd and } \omega'(\vec{C}_0) > 0 \\ \omega(\vec{C}_0) & \text{if } q \text{ is odd and } \omega'(\vec{C}_0) < 0 \end{cases}$$

which contradicts our assumption on \vec{C}_0 . This complete the proof of Step 1.

Step 2. (D, ω) is $(\leq q-1)$ -cycle-symmetric and contains no even cycles of length $k \in \{3, \dots, q-1\}$.

This follows from the choice of q and lemma 3.4.

Consider now the skew-signing ω'' of (D, ω) that coincides with ω' outside $\{e_1, e_1^*\}$ and such that $\omega''(e) = -\omega'_0(e)$ for $e \in \{e_1, e_1^*\}$. Let $p_{(D, \omega'_0)}(x) := x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ and $p_{(D, \omega'')}(x) := x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ be the characteristic polynomials of (D, ω'_0) and (D, ω'') respectively.

As in the proof of Lemma 3.4, we have

$$\begin{aligned} b_q - c_q &= -2(\eta_{\omega'_0}^+(e_1) + \eta_{\omega'_0}^+(e_1^*)) + 2(\eta_{\omega'_0}^-(e_1) + \eta_{\omega'_0}^-(e_1^*)) \\ &= -2(N_{\omega'_0}^+(e_1) - N_{\omega'_0}^-(e_1)) \neq 0 \end{aligned}$$

which contradicts Step 1. This ends the proof of Lemma. \square

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